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Stationary heat conduction in a solid with functionally graded thermal properties

Key words: stationary heat conduction, Fourier's law, functionally graded material, finite-difference method

Introduction

The aim of this paper is to find the temperature field in a solid made of an elastic material with functionally graded heat conductivity. Modelling the functionally graded materials has been investigated in many papers. Fundamentals of functionally graded materials were presented by Suresh and Mortensen (1998), later by Miyamoto, Kaysser, Rabin, Kawasaki and Ford (1999). Noda, Ootao and Tanigawa (2012) presented the solutions of transient thermoelastic problems for a functionally graded circular disk with piecewise power law. Olaijum, Hoe and Ogunbode (2017) published the paper where they present the use of finite-difference approximations to the heat equation. The finite-difference method was also used by Radzikowska and Wirowski (2012) for a two-dimensional heat conduction problem in the laminate with functionally graded properties. Laminates with functionally graded properties (Radzikowska & Wirowski, 2012) can be treated as made of functionally graded materials. The problems for functionally graded laminates or transversely graded laminates (Jędrysiak, 2011) are often solved by Tolerance Averaging Technique (Woźniak & Wierzbicki, 2000). In this paper the heat conduction problem in elastic, nonhomogeneous solid will be analysed.

Object of analysis

The object of analysis is an elastic, nonhomogeneous conductor made of a material with functionally graded heat properties. In this paper the problem of the two-dimensional, stationary heat conduction without heat sources will be solved. It will be assumed that heat conduction properties are changing only in one direction. The differential equation of this problem takes the form:

$$\frac{\partial}{\partial x}\left[k_1(x)\frac{\partial \vartheta}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_2\frac{\partial \vartheta}{\partial y}\right]=0 \tag{1}$$

where:
ϑ – unknown temperature field;
$k_1(x)$ – heat conductivity variable in direction x;
k_2 – constant heat conductivity in direction y.

A selected example will be a rectangular shape with side lengths of 1 and 1.5 m. Heat conductivity along short sides will mimic the properties of titanium alloy (Ti-6Al-4V) and zirconium oxide (ZrO_2) respectively. The materials were selected as an example of combination of two widely used media with varied mechanical properties – metallic alloys and ceramics (Noda et al., 2012). Heat conductivity coefficients for the mentioned materials are as follows, 6.2 $W \cdot m^{-1} \cdot K^{-1}$ for Ti-6AL-4V and 1.78 $W \cdot m^{-1} \cdot K^{-1}$ for ZrO_2, with the mean value of the two taken as an edge case along longer sides.

This paper will analyse three functions for the gradation of the length-wise heat conductivity. Functions were chosen is such a way that k_1 change would be linear, logarithmic, and exponential, according to equations (2)–(4) respectively.

$$k_1(x) = 0.0295x + 1.78 \tag{2}$$

$$k_1(x) = 0.615 \cdot \log_2 x + 1.78 \tag{3}$$

$$k_1(x) = 0.0043 \cdot 2^{x/15} + 1.78 \tag{4}$$

To solve the heat conduction equation (1) applied to the selected cases we will use the finite difference method with mesh size of 1 cm. Low computational complexity of the finite difference method allows for the use of dense meshing and easy validation of the developed model. For the considered differential equation its FDM temperature approximation at node $\vartheta_{i,j}$ is as follows

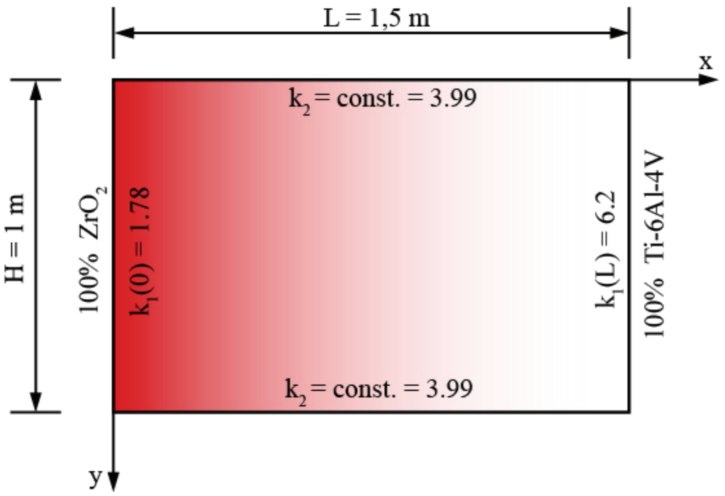

FIGURE 1. Conductor geometry and edge cases

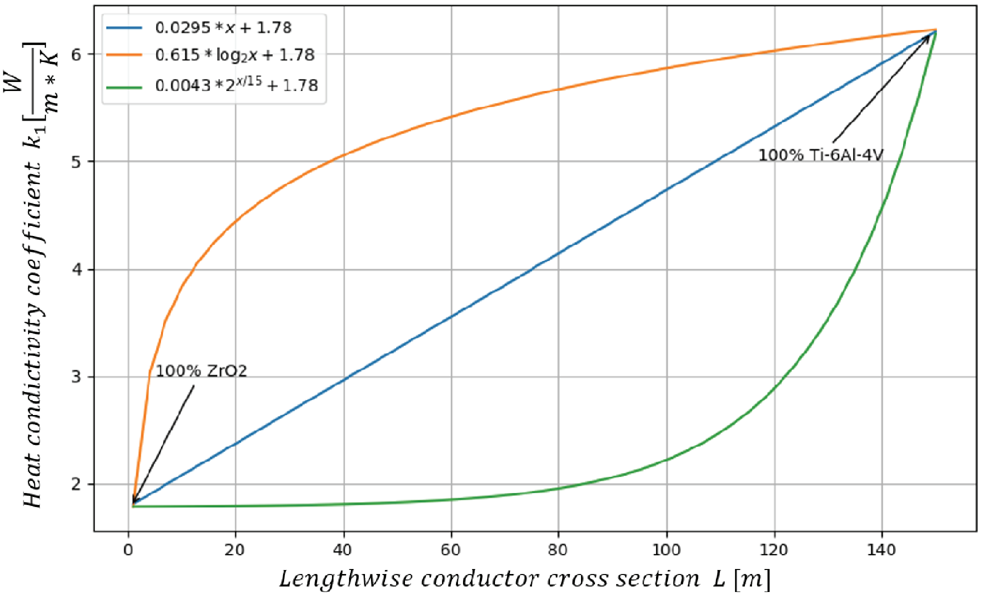

FIGURE 2. Heat conductivity gradation functions plot

$$A \cdot \vartheta_{i,j} + B \cdot \vartheta_{i-1,j} + C \cdot \vartheta_{i+1,j} + \\ + D \cdot \vartheta_{i,j-1} + E \cdot \vartheta_{i,j+1} = 0 \qquad (5)$$

where:

$$A = -2 \cdot k_1(i) - 2 \cdot k_2$$

$$B = k_1(i) + \frac{k_1(i-1) - k_1(i+1)}{4}$$

$$C = k_1(i) + \frac{k_1(i+1) - k_1(i-1)}{4}$$

$$D = E = k_2$$

Examples

Together with the aforementioned heat conductivity coefficient for the considered example we will assume the following Dirichlet's boundary conditions

$$\vartheta(0, y) = 500 \sin\left(\frac{\pi \cdot y}{H}\right), \ \vartheta(x, 0) = \\ = \vartheta(L, y) = \vartheta(x, H) = 0$$

The results of the analysis for each of the gradation functions are shown on Figures 4, 5 and 6 – for linear (2); on Figures 7, 8 and 9 – for logarithmic (3); on Figures 10, 11 and 12 – for exponential (4).

Conclusions

Figure 13 shows plots of the temperature distribution at cross sections $y = 0.5H$ for each of the analysed cases as well as the temperatures across conductors made solely of both titanium alloy and zirconium oxide.

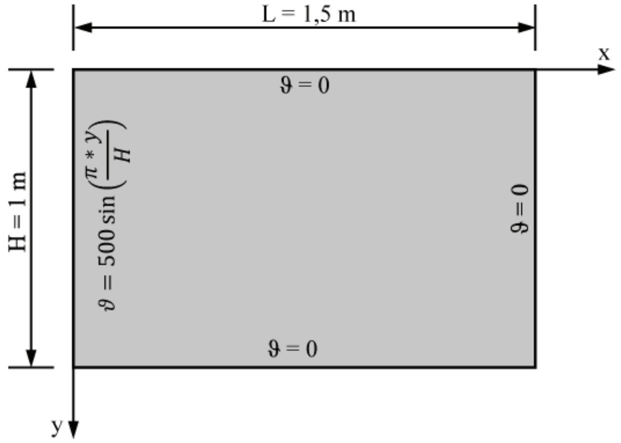

FIGURE 3. Boundary conditions of the model

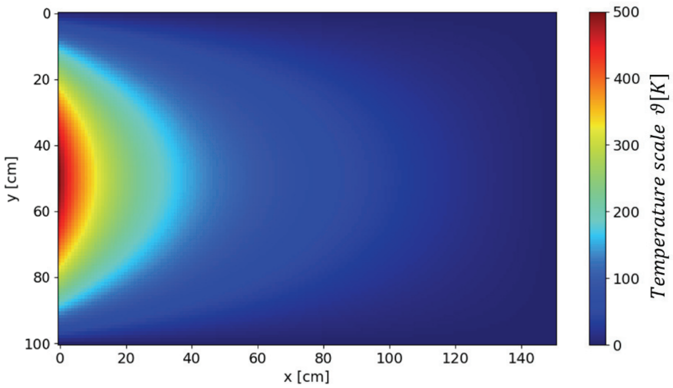

FIGURE 4. Temperature distribution with linear heat conductivity gradation – map

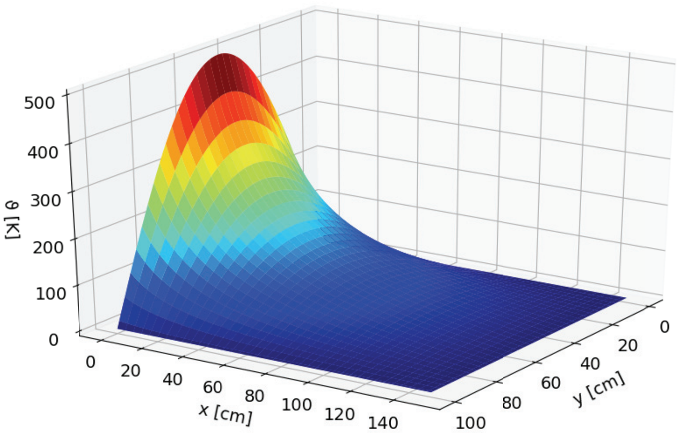

FIGURE 5. Temperature distribution with linear heat conductivity gradation – plot

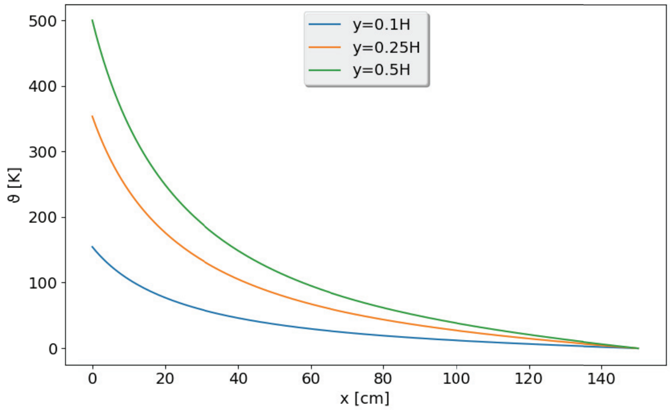

FIGURE 6. Temperature distribution – cross section at y = 0.1H, y = 0.25H and y = 0.5H

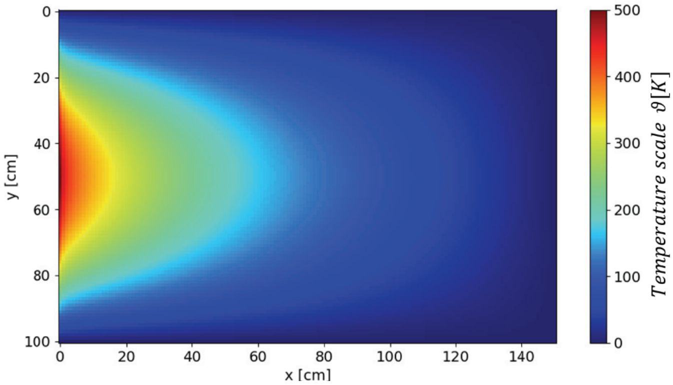

FIGURE 7. Temperature distribution with logarithmic heat conductivity gradation – map

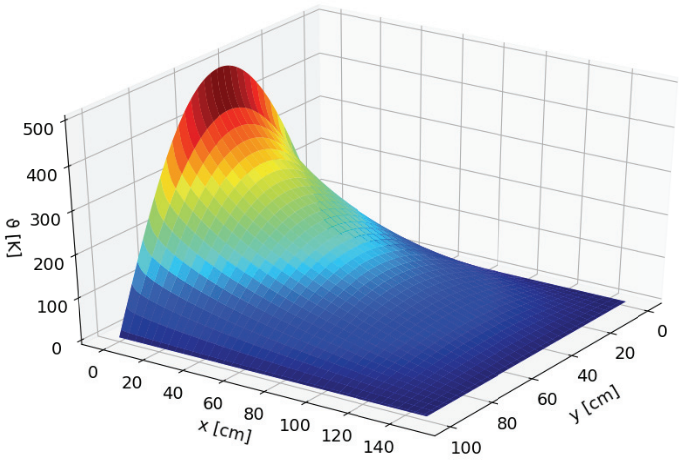

FIGURE 8. Temperature distribution with logarithmic heat conductivity gradation – plot

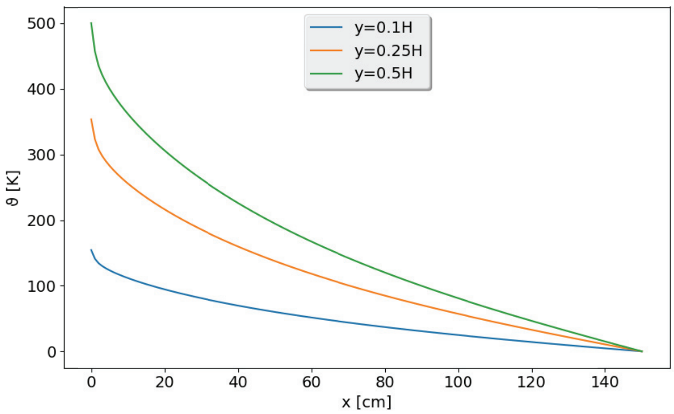

FIGURE 9. Temperature distribution – cross section at $y = 0.1H$, $y = 0.25H$ and $y = 0.5H$

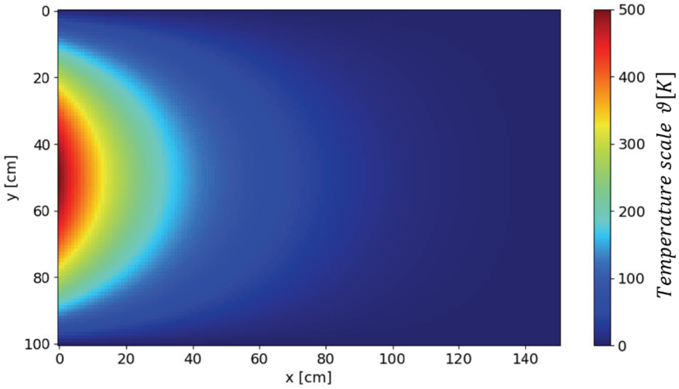

FIGURE 10. Temperature distribution with exponential heat conductivity gradation – map

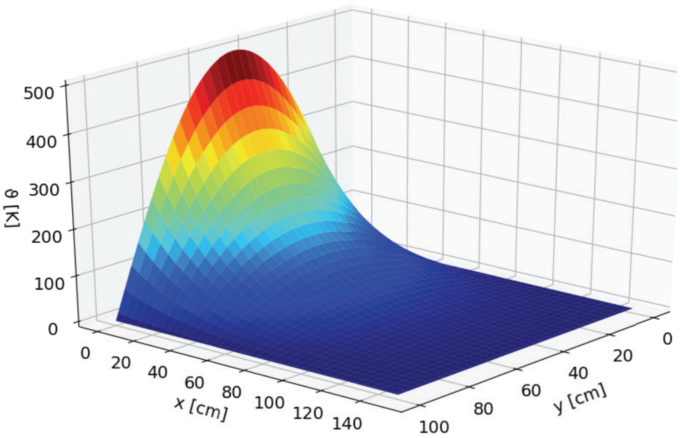

FIGURE 11. Temperature distribution with exponential heat conductivity gradation – plot

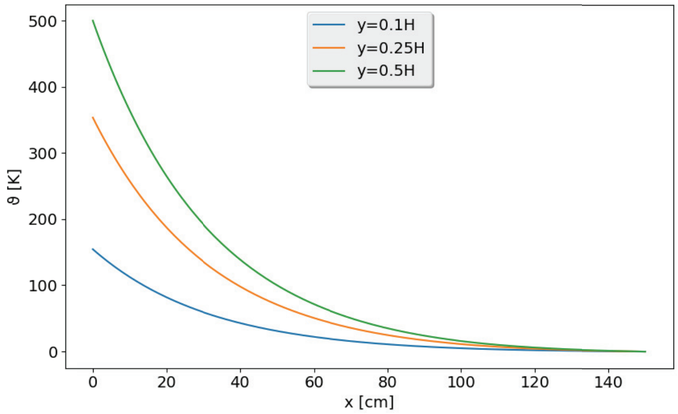

FIGURE 12. Temperature distribution – cross section at $y = 0.1H$, $y = 0.25H$ and $y = 0.5H$

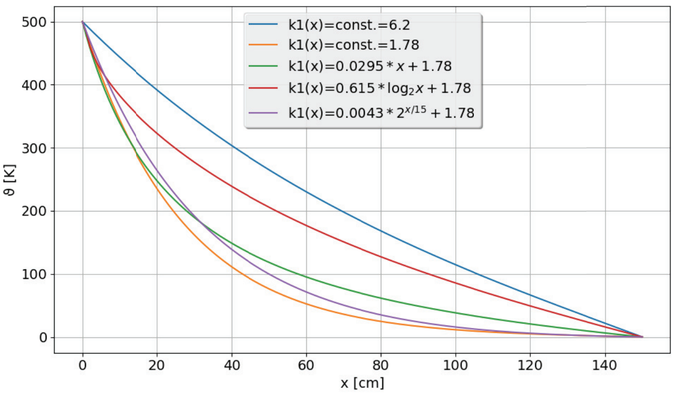

FIGURE 13. Comparison of temperature distribution in conductor at cross section $y = 0.5H$

Intuitively we can observe that in every case of FGM the temperature distribution falls between the homogeneous cases with linear and exponential cases crossing its values at 1/6 of the conductors length and exponential gradation showing the best insulating properties.

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Summary

Stationary heat conduction in a solid with functionally graded thermal properties. In the paper the solutions for stationary heat conduction in a two dimensional composite with functionally graded heat properties were obtained. Numerical solutions for the taken boundary conditions are shown for several types of changes of composite's thermal conductivity. The solutions were obtained with the use of the finite-difference method.

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